

An example of a quantum statistical model which cannot be mapped to a less informative one by any trace preserving positive map

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Comparison of statistical models (experiments) is an important branch of mathematical statistics, which gives deep insights in many aspects of foundation of statistics (see [5][7] and references therein, for example). Given a parameter space Θ (for simplicity, here we suppose Θ is a finite set), consider two families of probability distributions $\{p_\theta\}_{\theta \in \Theta}$ and $\{q_\theta\}_{\theta \in \Theta}$. Let \mathcal{D} be a space of the decision that a statistician may take, and for each $d \in \mathcal{D}$, let $l_\theta(d)$ be the loss when decision d is chosen and true parameter value is θ . We say the former is more informative than the latter, or

$$\{p_\theta\}_{\theta \in \Theta} \geq \{q_\theta\}_{\theta \in \Theta},$$

if and only if the following is true for all \mathcal{D} and $\{l_\theta\}_{\theta \in \Theta}$: for any the map d from the data space to the decision space, there is d' such that

$$\sum_x l_\theta(d'(x)) p_\theta(x) \leq \sum_x l_\theta(d(x)) q_\theta(x), \forall \theta \in \Theta.$$

Celebrated Brackwell theorem, or randomization criteria of LeCam [5][7] tells that this is equivalent to the existence of the transition probability $P(x|x')$

$$p_\theta(x) = \sum_{x'} P(x|x') q_\theta(x'), \forall \theta \in \Theta.$$

(For notational simplicity, here and below the set of all data (x 's) and the set where the decision takes value has finite number of elements. However, with proper mathematical settings, the above results essentially holds even if the former and the latter is an arbitrary measurable space and an arbitrary topological space, respectively.)

So far, there are two versions of quantum extensions of this concept [1][2][3][4]. One is to consider classical decision problem on quantum state families.

$$\{\rho_\theta\}_{\theta \in \Theta} \geq^c \{\sigma_\theta\}_{\theta \in \Theta}$$

if and only if the following is true for all \mathcal{D} and $\{l_\theta\}_{\theta \in \Theta}$: for any measurement $\{M_d\}_{d \in \mathcal{D}}$ taking values in \mathcal{D} , there is a measurement $\{M'_d\}_{d \in \mathcal{D}}$ such that

$$\sum_{d \in \mathcal{D}} l_\theta(d) \text{tr } M'_d \rho_\theta \leq \sum_{d \in \mathcal{D}} l_\theta(d) \text{tr } M_d \sigma_\theta, \forall \theta \in \Theta.$$

Another version is to consider the full quantum task. Consider a Hilbert space \mathcal{H}_D as an quantum analogue of decision space, and an operator L_θ in \mathcal{H}_D defining the loss. A decision rule is a completely positive trace preserving (CPTP) map Λ to operators on \mathcal{H}_D . Then we write

$$\{\rho_\theta\}_{\theta \in \Theta} \geq^q \{\sigma_\theta\}_{\theta \in \Theta}$$

if and only if the following is true for all \mathcal{H}_D and $\{L_\theta\}_{\theta \in \Theta}$: for any CPTP map Λ there is a CPTP map Λ' such that

$$\text{tr } L_\theta \Lambda'(\rho_\theta) \leq \text{tr } L_\theta \Lambda(\sigma_\theta), \forall \theta \in \Theta.$$

(Here, the loss measure is linear in the state. But, use of bounded and continuous functionals of the state over \mathcal{H}_D does not change the definition of \geq^q at all [4].)

" \geq^q " holds if and only if there is a CPTP map Γ such that [3][4]

$$\Gamma(\rho_\theta) = \sigma_\theta, \theta \in \Theta. \quad (1)$$

Meantime, if there is positive trace preserving Γ with above relation exists, then " \geq^c " holds obviously. A natural question is whether this is necessary. In this paper, we answer the question negatively by giving a counter example.

Let $\Theta = \{0, 1\}$, and consider the following condition,

$$\|\rho_0 - t\rho_1\|_1 \geq \|\sigma_0 - t\sigma_1\|_1, \forall t \geq 0, \quad (2)$$

where $\|A\|_1 := \text{tr} \sqrt{A^\dagger A}$. If $g(x)$ is a real valued function, $\|g\|_1 := \sum_x |g(x)|$.

We use the following lemma. This is not new [2][4], but the proof is stated for completeness.

Lemma 1 [2][4] Suppose $[\rho_0, \rho_1] = 0$. Then, $\{\rho_\theta\}_{\theta \in \Theta} \geq^c \{\sigma_\theta\}_{\theta \in \Theta}$ if and only if (2) holds.

Proof. Let $q_\theta^M(x) := \text{tr } \sigma_\theta M_x$, where $M = \{M_x\}$ is a POVM, $M_x \geq 0$, $\sum_x M_x = \mathbf{1}$. Then, by definition, $\{\rho_\theta\}_{\theta \in \Theta} \geq^c \{\sigma_\theta\}_{\theta \in \Theta}$ if and only if

$$\{\rho_\theta\}_{\theta \in \Theta} \geq \{q_\theta^M\}_{\theta \in \Theta}, \forall M.$$

This is equivalent to [6]

$$\|\rho_0 - t\rho_1\|_1 \geq \|q_0^M - tq_1^M\|_1, \forall M, \forall t \geq 0.$$

Therefore, since

$$\max_M \|q_0^M - tq_1^M\|_1 = \|\sigma_0 - t\sigma_1\|_1,$$

we have the assertion. ■

Below, we give an example such that $\{\rho_\theta\}_{\theta \in \Theta} \geq^c \{\sigma_\theta\}_{\theta \in \Theta}$ but there is no trace preserving positive map Γ with (1). The example is given as follows. Let

$$\rho_0 = \begin{bmatrix} \alpha & & \\ & 0 & \\ & & 1-\alpha \end{bmatrix}, \rho_1 = \begin{bmatrix} 0 & & \\ & \alpha & \\ & & 1-\alpha \end{bmatrix},$$

$$1 \geq \alpha \geq 0,$$

and

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \sigma_1 = \begin{bmatrix} 1-\beta^2 & \sqrt{1-\beta^2}\beta \\ \sqrt{1-\beta^2}\beta & \beta^2 \end{bmatrix}.$$

Further, we suppose

$$\alpha \geq \beta.$$

It is very easy to see that there is no trace preserving positive map Γ with (1). The proof runs as follows. Suppose there is such a positive map. Then for any pure state ψ_0 and ψ_1 in the support of ρ_0 and ρ_1 respectively, we should have

$$\Gamma(|\psi_\theta\rangle\langle\psi_\theta|) = \sigma_\theta, \theta \in \Theta.$$

However, $(001)^T$ is a common element of the support of ρ_0 and that of ρ_1 . Hence, with $\psi_0 = \psi_1 = (001)^T$, the above equation is impossible.

In addition, as will be shown in the following by elementary analysis, if holds, (2) is true. Hence, by the above lemma, this means $\{\rho_\theta\}_{\theta \in \Theta} \geq^c \{\sigma_\theta\}_{\theta \in \Theta}$ holds. Therefore, this is an example we need.

The proof is as follows. If $t < 0$, (2) holds obviously. Thus, with

$$f(t) := \|\rho_0 - t\rho_1\|_1^2 - \|\sigma_0 - t\sigma_1\|_1^2,$$

we prove $f(t) \geq 0$ for all $t \geq 0$.

If $t \geq 0$,

$$\|\rho_0 - t\rho_1\|_1 = \alpha(1+t) + (1-\alpha)|1-t|,$$

$$\|\sigma_0 - t\sigma_1\|_1 = \sqrt{(1-t)^2 + 4t\beta^2} \leq \sqrt{(1-t)^2 + 4t\alpha^2}$$

In the case of $0 \leq t \leq 1$,

$$f(t) \geq 4\alpha(1-\alpha)(-t^2+t) \geq 0.$$

In the case of $t \geq 1$,

$$f(t) \geq 4\alpha(1-\alpha)(t-1) \geq 0.$$

After all, we have $f(t) \geq 0$ for all $t \geq 0$.

References

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